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Longitudinal waves in partially saturated porous media: the effect of gas bubbles☆

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Abstract

The problem of the propagation of longitudinal waves in a liquid-saturated porous medium when there are gas bubbles present is considered. The decay factor and the phase velocity of Frenkel–Biot waves of the first and second kind are found as a function of the frequency in the linear approximation. It is shown that, in the neighbourhood of the resonance frequency of the bubbles, longitudinal Frenkel–Biot waves change their form. A wave of the first kind is transformed from a fast wave at low frequencies into a slow wave at high frequencies. The dispersion curve of a wave of the second kind consists of two branches – a "low-frequency" branch, the oscillations of which possess the classical properties, and a "high-frequency" branch, which is a weakly decaying high-velocity mode. The frequency dependences of the ratio of the mass velocities of a gas-liquid mixture and of a porous matrix, and also of the perturbations of the stress in the matrix and the pressure in the mixture, are constructed. It is shown that the "high-frequency" branch of a wave of the second kind is characterized by the in phase motion of the gas-liquid mixture and of the porous matrix, while their mass velocities are close, which explains the weak decay of this mode of oscillations. An analytical expression is obtained for the "boundary frequency", which determines the offset of the "high-frequency" branch of the dispersion curve of the wave of the second kind.

Longitudinal Frenkel–Biot waves of two kinds can propagate in porous media[1–3]: a wave of the first kind, due mainly to the compressibility of the intrapore filler and the material of the matrix, and a wave of the second kind, due mainly to the deformation of the matrix ("a matrix wave"). Their characteristics are determined by the difference in the densities and compressibilities of the component materials and the matrix.

A porous matrix can only be volume deformed if the saturating low-compressibility liquid is able to overflow in the system of pores, thereby freeing space for overpacking. The Darcy viscous resistance leads to strong decay, and hence a longitudinal Frenkel–Biot wave of the second kind corresponds to seismically observed waves in dry or almost dry soils and rocks,[4] where the viscous resistance of the gas (air) is negligible. In a medium completely saturated with liquid, the observed seismic wave becomes a longitudinal wave of the first king (fast), while a wave of the second kind can only propagate very short distances (due to the extremely strong decay). Consequently, it is logical to assume that for any intermediate value of the gas saturation, either a change in the type of observed longitudinal Frenkel–Biot wave occurs or one may simultaneously observe both types of longitudinal waves.[4,5]

In fact, in some publications[6–8] quite peculiar experimentally recorded graphs of the wave velocities against the frequency or other parameters of the medium have been presented, which can be interpreted as a transition from one

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observed type of wave to another. A number of researchers have assumed that in saturated porous media[5,9] or partially saturated porous media[3,10,11] different types of longitudinal Frenkel–Biot waves can propagate simultaneously.

The change in the behaviour of Frenkel–Biot waves of the first and second kind when gas bubbles are present has been observed by Bedford and Stern[12] and was then also investigated by other researches.[13,14] However, no explanation of the physical mechanism of the effect has been given. For example, it was stated directly in Ref. 12 that some of the results obtained may be entirely related to features of the numerical method and, consequently, make no physical sense.

In this paper we extend the investigation of the effect of a change in the behaviour of Frenkel–Biot waves when there is resonance of the gas bubbles. The change in the wave characteristics is due to a change in the nature of the motion of the liquid in the porous space when there are bubbles present. In fact, instead of overflow, which is necessary to obtain a longitudinal Frenkel–Biot wave of the second kind, the liquid may be displaced into the volume released when a bubble is compressed, allowing of the additional possibility of bulk deformation of the matrix. In this case the oscillations of the porous matrix and of the bubbles occur in phase, and the decay of a wave of the second kind (due to the reduction in the Darcy resistance) should be reduced considerably.

1. The equations of dynamics

Consider the propagation of longitudinal Frenkel–Biot waves in a liquid-saturated porous medium when there are gas bubbles present.

Suppose the wave propagates along the x axis. We will assume that the saturating liquid contains a gas in the form of identical isolated bubbles.

We will write the complete system of mass and momentum balance equations as follows:

$$\begin{aligned}
&\partial_t(1-m)\rho_1 + \partial_x(1-m)\upsilon_1\rho_1 = 0\\
&\partial_t m\rho_2 + \partial_x m\upsilon_2\rho_2 = 0\\
&\partial_t(1-m)\rho_1\upsilon_1 + \partial_x(1-m)\rho\upsilon_1^2 = \partial_x\sigma^f - (1-m)\partial_x p + \mu m^2 k^{-1}(\upsilon_2-\upsilon_1)\\
&\partial_t m\rho_2\upsilon_2 + \partial_x m\rho_2\upsilon_2^2 = -m\partial_x p - \mu m^2 k^{-1}(\upsilon_2-\upsilon_1)
\end{aligned} \tag{1.1}$$

We add to these equations the equation of the dynamics of a bubble[15]

$$R\partial_{tt}R + \frac{3}{2}(\partial_t R)^2 + \frac{4\mu}{\rho_L}\left(\frac{1}{R} + \frac{m}{4k}R\right)\partial_t R = (p_g - p)\rho_L^{-1} \tag{1.2}$$

The subscript 1 corresponds to the solid phase, the subscript 2 corresponds to the gas-liquid mixture, ρ_i and υ_i are the corresponding densities and mass velocities of the phases, ρ_L is the density of the liquid without the gas bubbles, $\sigma^f \equiv \sigma_{xx}^f$ is the effective Terzaghi stress,[3] p is the pressure in the liquid, m is the porosity, k is the permeability, μ is the viscosity of the liquid without the bubbles, R is the bubble radius, p_g is the gas pressure inside a bubble and $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_{tt} = \partial^2/\partial t^2$.

Note that, unlike the Rayleigh equation, which describes the oscillations of a single bubble in a viscous liquid, Eq. (1.2) is supplemented by a second dissipative term, which includes the porosity and permeability, in order to take into account the Darcy loss.

We will define the mean density of the gas-liquid mixture as

$$\rho_2 = \rho_L(1-\varphi), \quad \varphi = (4\pi/3)R^3 n_0 \tag{1.3}$$

where φ is the volume gas content and $n_0 = \text{const}$ is the number density of the bubbles. In the expression for the mean density (1.3) we have neglected the density of the gas, which is quite justified for low gas contents. The change in φ is due to the change in the bubble radius R, while their number density is assumed to be constant.

We will also assume that the pore pressure p is equal to the pressure in the liquid far from the bubble.

The system of equations is closed by the equations of state for the solid, liquid and gas phases, and also by the constitutive relation for the matrix

$$\rho_1 = \rho_{10}(1-\beta_1\sigma/3), \quad \rho_L = \rho_{L0}(1+\beta_L p), \quad p_g = p_0(R_0/R)^\chi$$
$$\sigma = I_1/(1-m)-3p, \quad \sigma^f = (K_b+4G/3)e_1 + K_b\beta_1 p \tag{1.4}$$

Here $\chi = 3\zeta$, ζ is the adiabatic exponent, K_b and β_1^{-1} are the moduli of volume elasticity of the porous matrix and of the material which constitutes the matrix, G is the shear modulus of the matrix, e_1 is the longitudinal strain of the matrix, β_L is the compressibility of the liquid and I_1 is the first invariant of the effective stress tensor.

Thermal effects, the added mass and the change in the liquid viscosity due to the presence of bubbles are ignored.

2. The linear approximation

We will assume that when a wave propagates the parameters of the medium deviate only slightly from those of its equilibrium state (we will denote the equilibrium parameters of the medium by a zero subscript):

$$m = m_0 + m^*, \quad p = p_0 + p^*, \quad \sigma^f = \sigma_0^f + \sigma^{f*}, \quad \upsilon_1 = \upsilon_1^*, \quad \upsilon_2 = \upsilon_2^*, \quad R^* = (R-R_0)/R_0$$
$$R^* \ll 1, \quad p^*/p_0 \ll 1, \quad m^*/m_0 \ll 1, \quad \upsilon_i^* \ll 1$$

System (1.1)–(1.4) can be reduced to the form

$$(1-m_0)(\rho_{10}\partial_{tt}u_1^* - \rho_{20}\partial_{tt}u_2^*) - \partial_x\sigma^{f*} = \mu m_0 k^{-1}(\partial_t u_2^* - \partial_t u_1^*)$$
$$\rho_{20}\partial_{tt}u_2^* + \partial_x p^* = -\mu m_0 k^{-1}(\partial_t u_2^* - \partial_t u_1^*)$$
$$\beta p^* - \beta_1\partial_x\sigma^{f*} - \frac{3\varphi_0}{1-\varphi_0}m_0 R^* + m_0\partial_x u_2^* + (1-m_0)\partial_x u_1^* = 0 \tag{2.1}$$
$$\partial_{tt}R^* + \omega_0^2 R^* + \xi\partial_t R^* = -\vartheta p^*$$
$$\sigma^{f*} = K_p\partial_x u_1^* + K_b\beta_1 p^*$$

Here

$$\beta = (1-m_0)\beta_1 + m_0\beta_L, \quad \omega_0^2 = \chi p_0/(\rho_{L0}R_0^2), \quad \xi = (4R_0^{-2} + m_0 k^{-1})\mu\rho_{L0}^{-1}$$
$$\vartheta = \rho_{L0}^{-1}R_0^{-2}, \quad K_p = K_b + 4G/3$$

where ω_0 is the resonance (Minnaert) frequency of a bubble in an unbounded liquid and u_1^* is the displacement of the i-th phase.

Below, in addition to the general case, we will also consider the case of "soft soil" ($\beta_1 K_b \ll 1$), when the last term in the constitutive relation (the last equation of (2.1)) and the second term in the third equation of (2.1) can be neglected.

To investigate the characteristic features of the propagation of longitudinal Frenkel–Biot waves in porous media we will seek a solution of system (2.1) in the form of a harmonic wave (the asterisk, which we placed on perturbations, will henceforth be omitted)

$$(u_1, u_2, p, R, \sigma^f) = (\delta u_1, \delta u_2, \delta p, \delta R, \delta\sigma^f)\exp[i(\omega t - \eta x)] \tag{2.2}$$

where ω is the frequency and η is the wave number.

Substituting expression (2.2) into system (2.1), we obtain a dispersion relation (the relation between the wave number and the frequency), starting from the condition for the solution of the system to be non-trivial, i.e. from the fact that the fifth-order determinant is equal to zero:

$$\det\|a_{mn}\| = 0 \tag{2.3}$$

where

$$a_{11} = -(1-m_0)\rho_{10}\omega^2 + i\omega\mu k^{-1}m_0^2, \quad a_{12} = (1-m_0)\rho_{20}\omega^2 - i\omega\mu k^{-1}m_0^2$$
$$a_{13} = 0, \quad a_{14} = 0, \quad a_{15} = i\eta$$

$$a_{21} = -i\omega\mu m_0 k^{-1}, \quad a_{22} = -\omega^2\rho_{20} + i\omega\mu m_0 k^{-1}, \quad a_{23} = -i\eta, \quad a_{24} = 0, \quad a_{25} = 0$$

$$a_{31} = -i\eta(1-m_0), \quad a_{32} = -i\eta m_0, \quad a_{33} = \beta, \quad a_{34} = -3\varphi_0 m_0/(1-\varphi_0), \quad a_{35} = i\eta\beta_1$$

$$a_{41} = 0, \quad a_{42} = 0, \quad a_{43} = \vartheta, \quad a_{44} = (-\omega^2 + \omega_0^2 + i\omega\xi), \quad a_{45} = 0$$

$$a_{51} = iK_p\eta, \quad a_{52} = 0, \quad a_{53} = -\varepsilon, \quad a_{54} = 0, \quad a_{55} = 1$$

Dispersion relations (2.3) can be represented as a quadratic equation

$$Z^2 + Z(M_1 + iM_2\tilde{\omega}) + M_3 + iM_4\tilde{\omega} = 0 \tag{2.4}$$

where

$$Z = c_m^2\frac{\eta^2}{\omega^2}, \quad \tilde{\omega} = \frac{\omega}{\omega_c}, \quad c_m^2 = \frac{K_p}{(1-m_0)\rho_{10}}, \quad \omega_c = \frac{m_0\mu}{k\rho_2} \tag{2.5}$$

and ω_c is the critical Biot frequency, related to the presence of the viscosity of the saturating liquid or gas.[2]

In the general case of a porous medium, saturated with a liquid with bubbles, the coefficients $M_1, \ldots, M_4$ depend on the gas content φ_0 and the babble radius. These expressions are not given here in view of their length.

Dispersion Eq. (2.4) has already been investigated by a number of researchers, for example in Refs. 1,3,11, but with simplifying assumptions. Thus, when there are no bubbles ($\varphi_0 = 0$ and $M_1 < 0$, $M_2 > 0$, $M_3 > 0$ and $M_4 < 0$) we can obtain well-known expressions[1,3,10] for the velocities of the fast and slow Frenkel–Biot waves. Another limiting case is the case of bubbles in an unbounded liquid.[16]

Eq. (2.4) has two solutions corresponding to Frenkel–Biot waves of the first and second kind.

3. Analytical solution when there is no viscosity

To investigate the effect of bubbles on the velocity of waves in a saturated porous medium, we will consider the special case when the saturating liquid is inviscid ($\mu = 0$). In this case it is possible to obtain a simple analytical expression for the wave velocities. In fact, with this assumption, for the class of "soft soils" ($\beta_1 K_b \ll 1$), system (2.1) reduces to a single equation, to which the following dispersion relation corresponds

$$(a_6 - a_3\omega^2)\eta^4 - (a_5 - a_2\omega^2)\omega^2\eta^2 + (a_4 - a_1\omega^2)\omega^4 = 0 \tag{3.1}$$

where

$$a_1 = (1-m_0)\beta\rho_{10}\rho_{20}R_0^2\rho_{L0}, \quad a_2 = g(\beta)R_0^2\rho_{L0}, \quad a_3 = K_p m_0 R_0^2\rho_{L0}$$

$$a_4 = (1-m_0)\bar{\beta}\rho_{10}\rho_{20}\chi p_0, \quad a_5 = g(\bar{\beta})\chi p_0, \quad a_6 = K_p m_0\chi p_0, \quad \bar{\beta} = \beta + \frac{3\varphi_0 m_0}{1-\varphi_0}(\chi p_0)^{-1}$$

$$\rho^* = m_0\rho_{10} + (1-m_0)\rho_{20}, \quad g(\beta) = (1-m_0)\rho^* + K_p\beta\rho_{20}$$

We then have the following expression for the phase velocity

$$V_\pm^2 = \frac{\omega}{\mathrm{Re}[\eta(\omega)]} = \frac{2(a_6 - a_3\omega^2)}{(a_5 - a_2\omega^2) \pm \sqrt{(a_5 - a_2\omega^2)^2 - 4(a_4 - a_1\omega^2)(a_6 - a_3\omega^2)}} \tag{3.2}$$

In the limiting case of very low frequencies

$$V_{\pm}^2\Big|_{\omega\to 0} = V_{0\pm}^2 = \frac{2a_6}{a_5 \pm \sqrt{a_5^2 - 4a_4a_6}} = \frac{2K_pm_0}{g(\bar\beta) \pm \sqrt{g^2(\bar\beta) - 4K_p\bar\beta m_0(1-m_0)\rho_{10}\rho_{20}}} \tag{3.3}$$

The total compressibility $\bar\beta$ now depends on the volume concentration of bubbles and the product χp_0. For the Frenkel–Biot theory (a saturated porous medium without bubbles, $\varphi_0 = 0$) the coefficient $\bar\beta = \beta$.

We will determine what type of wave each solution characterizes. We recall that when $\varphi_0 = 0$ it is generally accepted to call the slow wave a wave of the second kind, and the fast wave a wave of the first kind. We will retain the name "wave of the second kind" for a wave that is slow in the low-frequency region ($\omega \ll \omega_0$), and we will call the fast wave a wave of the "first kind". Hence, solution (3.3) with a plus sign corresponds to a wave of the second kind, while the solution with a minus sign corresponds to a wave of the first kind.

In the case of high frequencies, the phase velocities tend to other limit values

$$V_{\pm}^2\Big|_{\omega\to \infty} = V_{\infty\pm}^2 = \frac{-2a_3}{-a_2 \pm \sqrt{a_2^2 - 4a_1a_3}} = \frac{-2K_pm_0}{-g(\beta) \pm \sqrt{g^2(\beta) - 4K_p\beta m_0(1-m_0)\rho_{10}\rho_{20}}} \tag{3.4}$$

Now the plus sign in expression (3.4) corresponds to the fast wave, while the minus sign corresponds to the slow wave. Hence, we can assume that at some frequency a cardinal change in the wave velocities has occurred – the initially faster wave has become the slow wave while the slow wave has become the fast wave. Note that even for a low but finite gas content the asymptotic values (3.3) and (3.4) of the phase velocities are in no way identical due to the difference $\bar\beta = \beta$.

Before proceeding with the analysis, we will consider the limiting case, which has been well investigated, of a liquid with bubbles (when there is no porous medium $K_p = 0$ and $m_0 = 1$). The coefficients a_6 and a_3 will then be equal to zero, and Eq. (3.1) will only have one root

$$V^2 = \frac{\rho^*(\omega_0^2 - \omega^2)}{\rho_{10}\rho_{20}\beta(\omega_{gL}^2 - \omega^2)}; \quad \omega_{gL}^2 = \omega_0^2\frac{\bar\beta_L}{\beta_L}, \quad \bar\beta_L = \beta_L + \frac{3\varphi_0}{1-\varphi_0}(\chi p_0)^{-1} \tag{3.5}$$

It obviously follows from expression (3.5) that there are two branches of the oscillations – a "low-frequency" branch ($\omega < \omega_0$) and a "high-frequency" branch ($\omega > \omega_{gL}$). The frequency range $\omega_0 < \omega < \omega_{gL}$ corresponds to a non-transparency window – in this frequency band when there is no viscosity the square of the phase velocity becomes negative and a wave cannot propagate.

We will now return to our investigation of dispersion relation (3.1). We draw attention to the fact that the function $\eta = \eta(\omega)$ intersects the abscissa axis at two points: at zero frequency ($\omega = 0$) and at a frequency $\omega_g = a_4/a_1$. The point $\omega = 0$ is the origin of the low-frequency branch, while the point ω_g is the origin of the high-frequency branch

$$\omega_g^2 = \omega_0^2\frac{\bar\beta}{\beta} \tag{3.6}$$

Unlike the case of a liquid with bubbles (the second relation of (3.5)) the boundary frequency now depends on the porosity.

All the calculations presented below were carried out for the following parameters: the porosity $m_0 = 0.25$, the compressibility of the liquid $\beta_2 = 2\times10^{-9}\,\text{Pa}^{-1}$, the compressibility of the material of the porous matrix $\beta_1 = 2\times10^{-10}\,\text{Pa}^{-1}$, the bulk modulus of elasticity of the porous matrix $K_b = 5\times10^7$ Pa, the density of the liquid $\rho_{20} = 1000\,\text{kg/m}^3$, the density of the material of the porous matrix $\rho_{10} = 2500\,\text{kg/m}^3$, the steady pressure $p_0 = 10^5$ Pa, the initial bubble radius $R_0 = 5\times10^{-5}$ m, and the permeability $k = 2\times10^{-11}\,\text{m}^2$.

In Figs. 1 and 2 we show graphs of the phase velocity V of the Frenkel–Biot wave of the first kind (the dashed curves) and of the second kind (the continuous curves), and also the wave number $\text{Re}\eta$ and the attenuation coefficient $\Delta = -\text{Im}\eta$ as a function of the normalized frequency $\Omega = \omega/\omega_0$ ignoring dissipation ($\mu = 0$) for a gas content $\varphi_0 = 10^{-4}$ (curves 1) and $\varphi_0 = 10^{-3}$ (curves 2).

The bubble phase (even at a low gas content) turns out to have a strong effect on the nature of the wave process, namely the dispersion curve of the Frenkel–Biot wave of the second kind has two oscillation branches (Fig. 2) – a

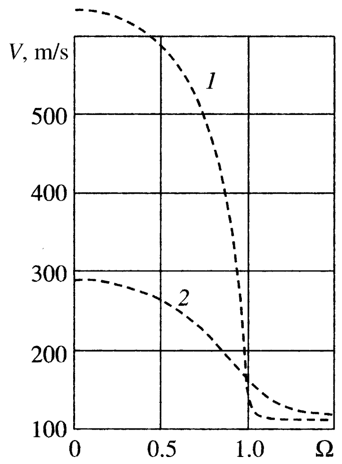

Fig. 1.

low-frequency branch when $\omega < \omega_0$ and a high-frequency branch when $\omega > \omega_g$ (ω_g is the boundary frequency, beginning from which the high-frequency branch of the oscillations appears, see formula (3.6)).

The "window of instability" of a wave of the second kind corresponds to the frequency band $\omega_0 < \omega < \omega_g$ when there is no dissipation: here the square of the wave number η is less than zero, while $\mathrm{Im}\eta > 0$ (Fig. 2). For a wave of the first kind the decay factor $\Delta = 0$ over the whole frequency range.

It is also important that, in the region of the resonance frequency of oscillations of the bubble radius, a change in the wave characteristics occurs: the wave of the first kind becomes a slow wave and conversely the velocity of the wave of the second kind increases considerably (compare with[14]).

The dynamics of waves in instability windows was investigated previously in Refs. 17–20, but using other equations[21,22] for a viscoelastic medium with oscillating solid fragments (granules).

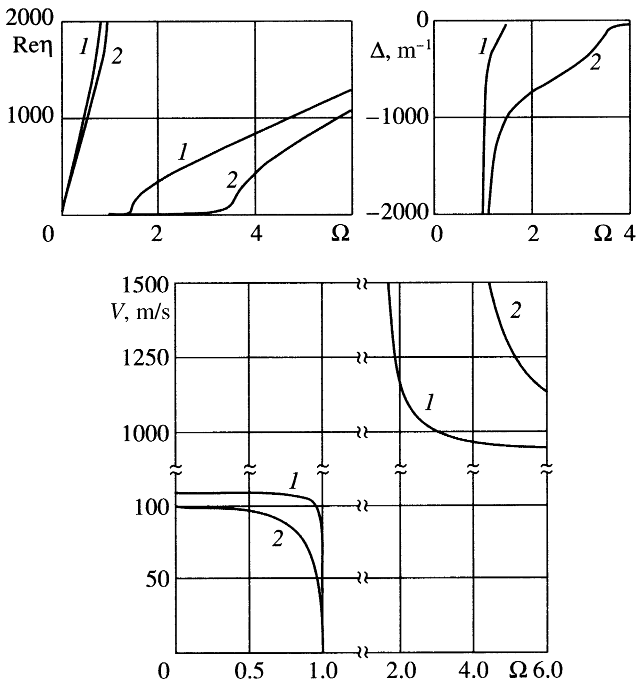

Fig. 2.

4. Numerical investigation of the dispersion curves taking viscosity into account

From the physical point of view, the high-frequency branch of the dispersion curve of a Frenkel–Biot wave of the second kind is characterized by antiphase oscillations of the pressure in the liquid and of the bubble radius.[13] We will consider this problem in more detail.

We will introduce the effective compressibility β_g of the bubbles,[13] and the compressibility of the gas-liquid mixture (U_g is the bubble volume)

$$\beta_g = -\frac{1}{U_g}\frac{\partial U_g}{\partial p}; \quad \beta_\Sigma = (1-\varphi_0)\beta_L + \varphi_0\beta_g \tag{4.1}$$

Suppose the pressure in the liquid and the bubble radius vary harmonically [2.2]. Then, for a viscous liquid ($\mu \neq 0$) it follows from the first equation of (4.1) and the last equation of (2.1) that the compressibility of the gas is a complex quantity

$$\beta_g = 3[-R_0^2\rho_{L0}\omega^2 + \chi p_0 + i\mu(4 + mk^{-1}R_0^2)\omega]^{-1} \tag{4.2}$$

Then $\mathrm{Im}\beta_g$ defines the phase difference between the variations of the pressure and the bubble radius.[11]

Graphs of the absolute value of the compressibility ($|\beta_g|$) and of the phase difference Ψ_g against the normalized frequency $\Omega = \omega/\omega_0$ are shown in Fig. 3 for $\mu = 10^{-3}$ Pa. Whereas at low frequencies ($\omega \ll \omega_0$) the change in the pressure and in the bubble radius were in phase, at high frequencies ($\omega \gg \omega_0$) they were out of phase (Fig. 3b); at high frequencies, when the pressure in the liquid increases, the bubble expands, which also changes the oscillation mode. It can be seen that at high frequencies the absolute value of the compressibility of the gas-liquid mixture ($|\beta_\Sigma|$) approaches the compressibility of the liquid alone (Fig. 4a). On the graph of the frequency dependence of the phase shift (Ψ_Σ) of the variations of the density of the gas-liquid mixture and the pressure, the interval of out-of-phase oscillations (Fig. 4b) corresponds to a transition zone between the low-frequency and high-frequency branches.

For the values of the parameters indicated in Section 3, we show in Figs. 5 and 6 graphs of the phase velocity V of Frenkel–Biot waves of the first kind (the dashed curves) and of the second kind (the continuous curves) and of the decay factor $\Delta = -\mathrm{Im}\eta$ against the normalized frequency $\Omega = \omega/\omega_0$, taking the dissipation into account ($\mu = 10^{-3}$ Pa s) for a gas content $\varphi_0 = 10^{-4}$ (curves 1) and $\varphi_0 = 10^{-3}$ (curves 2).

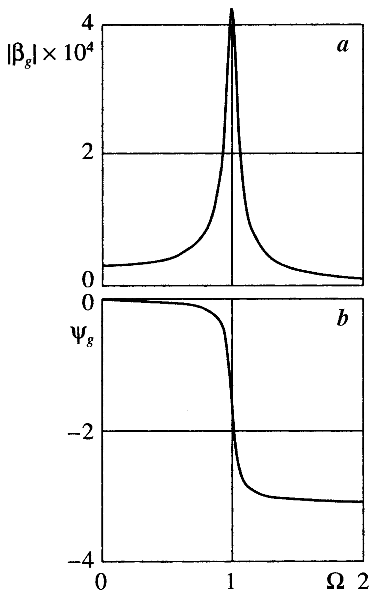

Fig. 3.

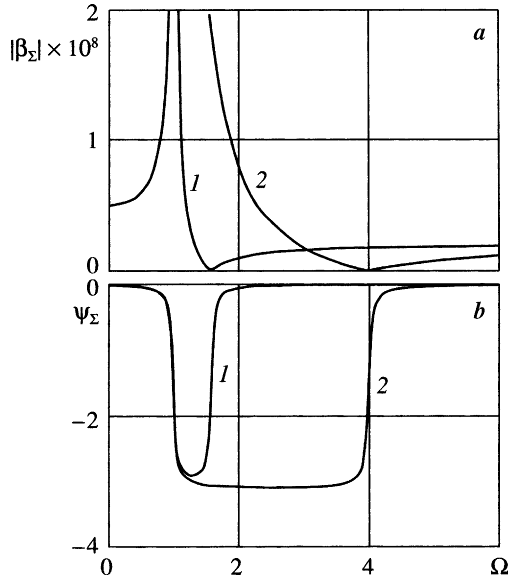

Fig. 4.

As follows from the calculations, even at a low viscosity the instability of the Frenkel–Biot wave of the second kind in the region $\omega_0 < \omega < \omega_g$ is suppressed, and both its dispersion curves join up (Fig. 6a). It is important to emphasise that the high-frequency branch of the dispersion curve of a wave of the second kind corresponds to rapidly propagating and weakly decaying waves: it is precisely these oscillations which can be recorded experimentally (Fig. 6).

When there are bubbles present a wave of the first kind is transformed from a fast wave at low frequencies into a slow wave at high frequencies (Fig. 5b), and its decay factor at low frequencies is much less ($\omega < \omega_0$) than at high

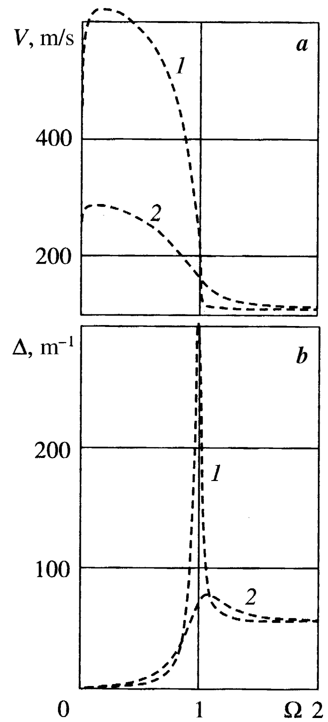

Fig. 5.

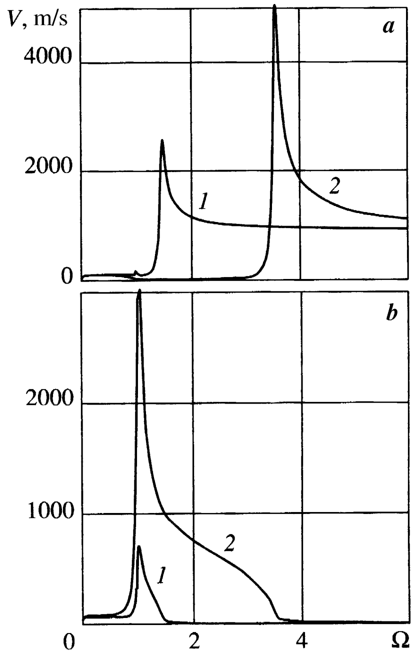

Fig. 6.

frequencies ($\omega > \omega_g$). Maximum absorption of the energy of the wave occurs at the resonance frequency of the bubbles (Fig. 5a).

We will consider the possibility of changing the type of relative motion of the gas-liquid mixture and the porous matrix for Frenkel–Biot waves at the resonance frequency. As is well known,[3] a wave of the first kind is characterized by in phase motion of the liquid and of the porous matrix, while a wave of the second kind is characterized by out-of-phase motion. If in the region of the resonance frequency of the oscillations of the bubble radius, a change in the phase shift between the motion of the liquid and of the porous matrix occurs together with a change in the velocities and decay of the waves (the in phase motion changes into out-of-phase motion and vice versa), we can speak of a change in the type of relative motion in the Frenkel–Biot waves.

In the linear approximation, the system of Eqs. (1.1)–(1.4) enables us to calculate the ratio of the mass velocities of the gas-liquid mixture v_2 and the porous matrix v_1, and also the ratio of the perturbations of the stress in the matrix and the pressure in the gas-liquid mixture

$$\frac{v_2}{v_1} = \frac{J_1}{J_2}, \quad J_1 = \begin{vmatrix} 1-\Pi & \rho_{10}(1-m_0) \\ \rho_{20}^{-1} & i\theta_c \end{vmatrix}, \quad J_2 = \begin{vmatrix} \rho_{20}m_0 & 1-\Pi \\ 1-i\theta_c & \rho_{20}^{-1} \end{vmatrix}, \quad \frac{\sigma^f}{p} = \Pi$$

$$\Pi = \frac{\varepsilon + (1-i\theta_c - m_0)\zeta}{1+(1-i\theta_c)\zeta}, \quad \zeta = \frac{\rho_{10}Z}{\rho_{20}\Xi}, \quad \Xi = \begin{vmatrix} \rho_{20}m_0 & \rho_{10}(1-m_0) \\ 1-i\theta_c & i\theta_c \end{vmatrix}, \quad \theta_c = \frac{\omega_c}{\omega_0}$$

The quanity Z is defined above (see formula (2.5)).

Unlike the previous calculations, we will henceforth consider a "cementized" geo material ($K_b\beta_1 \approx 0.5$), in which the effective stresses are considerable. The remaining parameters of the problem are given in Section 3 and remain unchanged; $\mu = 10^{-3}$ Pa s and $\varphi_0 = 10^{-4}$.

In Figs. 7 and 8 we show graphs of the absolute value of the ratio of the mass velocities v_2/v_1 of the gas-liquid mixture and of the porous matrix, and also of the phase shift Ψ between the displacements of the gas-liquid mixture and the porous matrix (in degrees) against the normalized frequency $\Omega = \omega/\omega_0$ for Frenkel–Biot waves of the first kind (the dashed curves) and of the second kind (the continuous curves).

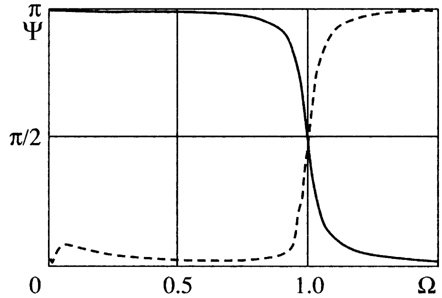

Fig. 7.

At low frequencies of the Frenkel–Biot wave of the second kind, the gas-liquid mixture and the porous medium move out of phase (Fig. 7), but at frequencies exceeding ω_g, the out-of-phase motion changes into in phase motion (which also explains the weak decay of the high-frequency branch.

Moreover, whereas at low frequencies the mass velocity (in absolute value) of the gas-liquid mixture v_2 exceeds the mass velocity of the porous matrix v_1, at high frequencies their mass velocities are close to one another (Fig. 8b).

For a Frenkel–Biot wave of the first kind, the opposite situation is observed (Figs. 7 and 8a), namely, at high frequencies the in phase motion changes to out-of-phase motion, and the mass velocity of the gas-liquid mixture becomes ten times greater than the mass velocity of the porous matrix.

Fig. 9 enables us to follow the frequency dependence of the absolute value of the ratio of the perturbations of the effective stress σ^f and the pressure p in Frenkel–Biot waves of the first kind (the dashed curves) and the second kind (the continuous curves) for a gas content $\varphi_0 = 10^{-4}$. As previously, the normalized frequency $\Omega = \omega/\omega_0$ is used in the graphs. Whereas at low frequencies of the wave of the first kind $|\sigma^f| \sim 0.6|p|$ (Fig. 9a), while in the case of the wave of

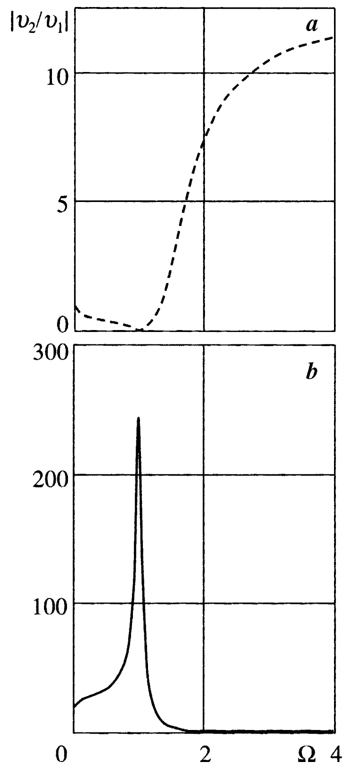

Fig. 8.

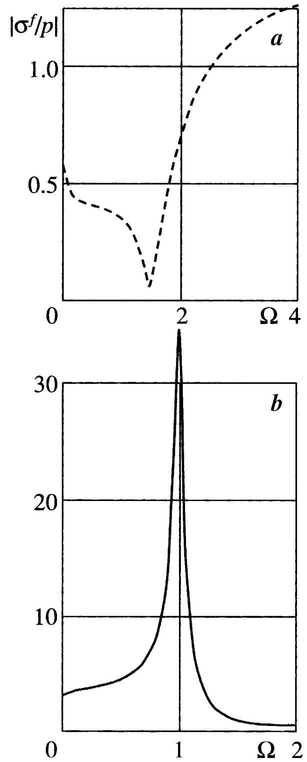

Fig. 9.

the second kind $|\sigma^f| \sim 4|p|$, in the high-frequency region the opposite situation is observed. Note that for a Frenkel–Biot wave of the second kind (Fig. 9b) the maximum of the absolute value of the ratio of the perturbations of the effective stress σ^f and the pressure p is reached at the resonance frequency of the bubbles, since at the resonance frequency of the bubbles, due to the increase in the compressibility of the liquid, the pressure value falls sharply. The absolute value of the ratio of the mass velocities v_2/v_1 is also high ($|v_2/v_1| \approx 250$).

A Frenkel–Biot wave of the first kind, conversely, in the neighbourhood of the resonance frequency is characterized by an increase in the pressure ($|\sigma^f/p| \approx 0.05$, Fig. 9a).

Hence, at the resonance frequency of the bubbles, first, a change in the type of relative motion of the gas-liquid mixture – porous matrix occurs in Frenkel–Biot waves of the first and second kinds (the in phase motion of the porous matrix and the gas-liquid mixture changes into out-of-phase motion and vice versa). Second, the ratio of the absolute values of the mass velocities of the gas-liquid mixture and of the porous matrix changes (whereas at low frequencies displacements of the matrix predominate, at high frequencies the displacement of the gas-liquid mixture will be greater and vice versa). As a consequence the ratio of the values of the effective stress σ^f and the pressure p also changes. The Frenkel–Biot waves of the first and second kind "change their roles" in their action on the matrix.

Note that, from the practical point of view, the wave with out-of-phase motion in the saturating liquid (or gas) – porous matrix is important, since it is precisely the out-of-phase nature of the wave that enables the porous channels to be cleaned. The Frenkel–Biot wave of the second kind in the porous medium saturated with liquid without gas bubbles is characterized by such an out-of-phase motion and a predominance of the effective stress over the pressure.

However, as noted above, when bubbles are present in the region of the resonance frequency a change in the Frenkel–Biot waves occurs. At high frequencies the out-of-phase motion is already present in the wave of the first kind, and the effective stress exceeds the pressure ($|\sigma^f| \sim 1.4|p|$). Simultaneously, its decay increases considerably. The high-frequency branch of the dispersion curve of the Frenkel–Biot wave of the second kind is characterized by in phase motion in the gas-liquid mixture – porous matrix.

5. Conclusions

We have given a physical interpretation of the effect of the trans-formation of Frenkel–Biot waves of the first and second kind when gas bubbles are present.

The change-over of the wave characteristics is due to a change in the compressibility of the gas-liquid mixture, namely its sharp increase at the resonance frequency of the bubbles. Physically this means that whereas before resonance the "fast" wave was due mainly to the compressibility of the gas-liquid mixture, at resonance it will be determined mainly by the deformation of the porous matrix, since its compressibility becomes higher than the compressibility of the gas-liquid mixture. Mathematically this corresponds to a change in the sign in the dispersion equation.

It has been shown that when gas bubbles are present in the saturated porous medium, the dispersion curve of the Frenkel–Biot wave of the second kind consists of two branches – a "low-frequency" branch and a "high-frequency" branch. These branches, when there is no dissipation, are separated by a non-transparency window – the band between the resonance and boundary frequencies. The distinguishing feature of the high-frequency branch is the fact that it describes rapidly propagating and weakly decaying waves: it is these oscillations that can be recorded experimentally. The decay of such waves is less but the propagation velocity is higher than for a wave of the first kind. Conversely, at frequencies exceeding the boundary frequency, the wave of the first kind is slowed down and its decay increases.

By an analytical investigation of the parameters of the waves when a porous medium is saturated with an inviscid liquid with gas bubbles we have obtained expressions for the asymptotic values of the phase velocities of Frenkel–Biot waves in the low-frequency and high-frequency limit, and also for the boundary frequency (the frequency, beginning from which a high-frequency branch of the oscillations of the wave of the second kind appears).

By means of numerical calculations we have found that, in the neighbourhood of the resonance frequency, the character of relative motion between the gas-liquid mixture and the porous matrix changes in both types of longitudinal waves. As a consequence, the ratio of the values of the effective stress σ^f and the pressure p also changes. Waves of the first and second kind "change roles" in their action on the matrix.

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